



TITLE:

Remarks on the cuspidal simple module of  $GL(p,q)$  over a field of characteristic  $p$  with  $p \mid q-1$  (Cohomology theory of finite groups and related topics)

AUTHOR(S):

奥山, 哲郎

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# Remarks on the cuspidal simple module of $GL(p, q)$ over a field of characteristic $p$ with $p \mid q - 1$

北海道教育大学・旭川校 奥山 哲郎

Tetsuro Okuyama

Hokkaido University of Education, Asahikawa Campus

## 1 Introduction and Results

Let  $p$  be an odd prime and  $k$  be an algebraically closed field of characteristic  $p$ . Let  $q$  be a prime power with  $p \mid q - 1$  and set  $\tilde{G} = GL(p, q)$ ,  $G = \tilde{G}/Z(\tilde{G}) = PGL(p, q)$ . The principal block algebra  $B_0(k\tilde{G})$  has a unique cuspidal simple module (of dimension  $\prod_{i=1}^{p-1}(q^i - 1)$ ) and it is known to be periodic as a  $kG$ -module. Our purpose in this talk is to describe the support variety of the cuspidal simple module. For this, we apply the so called a cohomological pushout method of constructing endotrivial modules developed by Carlson and Thévenaz who classified such modules for  $p$ -groups. For the method refer the articles by Carlson [8], by Carlson, Mazza and Thévenaz [10] and see the references in the article on classification theorem.

### 1.1 $p$ -Local subgroups of $GL(p, q)$

Let  $p^n$  be the exact power of  $p$  dividing  $q - 1$ . Then a Sylow  $p$ -subgroup  $\tilde{P}$  of  $\tilde{G}$  is isomorphic to  $\mathbb{Z}_{p^n} \wr \mathbb{Z}_p$ . Write  $\tilde{P} = \tilde{Q} \rtimes \langle \tilde{a} \rangle$  where  $\tilde{Q}$  is a Sylow  $p$ -subgroup of the group of diagonal matrices in  $\tilde{G}$  and  $\tilde{a}$  is a permutation matrix corresponding to a suitable cyclic permutation of length  $p$ . We know that  $Z_2(\tilde{P}) \cong \langle \tilde{b} \rangle \times Z(\tilde{P})$  for some element  $\tilde{b} \in \tilde{Q}$  of order  $p$  and  $\tilde{E} := \langle \tilde{a}, \tilde{b} \rangle \cong p_+^{1+2}$ . We have

$$N_{\tilde{G}}(\tilde{Q}) = \tilde{D} \rtimes \tilde{W} \quad (1.1)$$

where  $\tilde{D}$  is the group of diagonal matrices in  $\tilde{G}$  and  $\tilde{W} \cong \Sigma_p$  is the group of permutation matrices of degree  $p$ , the Weyl group of  $\tilde{G}$ . Set  $\tilde{A} = \langle \tilde{a} \rangle$ . Then  $\tilde{A} \subset \tilde{W}$  and  $N_{\tilde{W}}(\tilde{A}) = \tilde{H} \rtimes \tilde{A}$  for some subgroup  $\tilde{H} \subset \Sigma_{p-1}$  such that  $\tilde{H} \cong \mathbb{Z}_{p-1}$ . And we have

$$N_{\tilde{G}}(\tilde{P}) = Z(\tilde{G})(\tilde{Q} \rtimes (\tilde{A} \rtimes \tilde{H})) \subset N_{\tilde{G}}(\tilde{Q}) \quad (1.2)$$

$N_{\tilde{G}}(\tilde{E})$  has a subgroup  $\tilde{L}$  such that  $\tilde{L} \cong SL(2, p)$  and

$$N_{\tilde{G}}(\tilde{E}) = Z(\tilde{G}) * (\tilde{E} \rtimes \tilde{L}) \quad (1.3)$$

Let  $P$ ,  $Q$  and  $E$  be the images in the factor group  $G = \tilde{G}/Z(\tilde{G}) = PGL(p, q)$  of  $\tilde{P}$ ,  $\tilde{Q}$  and  $\tilde{E}$ , respectively.  $P$  is a Sylow  $p$ -subgroup of  $G$ .

$E = \langle a, b \rangle$  is an (maximal) elementary abelian  $p$ -subgroup of  $G$  of rank 2 where  $a$  and  $b$  are images in  $G$  of  $\tilde{a}$  and  $\tilde{b}$ , respectively. Note that  $Q$  is of index  $p$  in  $P$ .  $N_G(Q)$  and  $N_G(E)$  (and  $N_G(P)$ ) controll the  $p$ -fusion in  $G$ . For these facts, see the paper by Alperin and Fong [2]. We shall give the elements  $a$  and  $b$  concretely, below.

As we denote by  $P$  the image of  $\tilde{P}$  in  $G$ , in the following we shall denote the images of various subgroups  $\tilde{K}$  of  $\tilde{G}$  in  $G$  by  $K$ .

## 1.2 The cuspidal simple module

The Weyl group  $W$  is isomorphic to  $\Sigma_p$  which is a Coxeter group of type  $A_{p-1}$ . For a subset  $J$  of the generating set for  $W$ , let  $\tilde{G}_J$  be the corresponding standard parabolic subgroup of  $\tilde{G}$ . And the set of parabolic subgroups defines so called the Tits Building for  $\tilde{G} = GL(p, q)$ . Associated with the building, we have a complex of  $k\tilde{G}$ -modules (actually, of  $kG$ -modules) of the following form ;

$$\cdots \rightarrow 0 \rightarrow X^{p-2} \xrightarrow{f} X^{p-3} \rightarrow \cdots \rightarrow X^0 \rightarrow k_G \rightarrow 0 \rightarrow \cdots \quad (1.4)$$

where  $X^i = \bigoplus_{|J|=p-2-i} k_{\tilde{G}_J} \uparrow^{\tilde{G}}$ . In particular,  $X^{p-2} = k_{\tilde{B}_0} \uparrow^{\tilde{G}}$  where  $\tilde{B}_0$  is a Borel subgroup of  $\tilde{G}$ .

In our setting that  $q-1 \equiv 0 \pmod{p}$ , we have the following situation.

The sequence is exact except at  $X^{p-2}$ . Each term  $X^i$  is a  $Q$ -projective permutation  $kG$ -module and the sequence is  $Q$ -split. In particular, the sequence is the first  $p-1$  terms of a  $Q$ -projective resolution of  $k_G$  (although, terms are not minimal). This fact is shown by , for example, by an investigation by Cabanes and Rickard [7]. The Brauer character of  $\text{Ker } f$  is a modular reduction of the Steinberg character. By results of James (Lemma 3.4 [13]) and Geck, Hiss and Malle (Theorem 4.2 [12]),  $\text{Top Ker } f$  is a cuspidal simple  $kG$ -module whose Brauer character is the modular reduction of an ordinary cuspidal irreducible character (of degree  $\prod_{i=1}^{p-1} (q^i - 1)$ .) and projective when restricted to  $Q$ . For these facts, see also a study by Geck [11].

Set  $S = \text{Top Ker } f$ , the cuspidal simple  $kG$ -module.

## 1.3 A $Q$ -projective resolution of $k_G$

The principal block  $B_0(kG)$  has a close relation with the group algebra  $k\Sigma_p$  (and with the Hecke algebra  $\text{End}_{kG}(k_{B_0} \uparrow^G)$ ). A minimal  $Q$ -projective resolution of  $k_G$  is described as follows.

By studies of Dipper, James and others (see [11]), we can label the set of the projective indecomposable module of the Hecke algebra  $\text{End}_{kG}(k_{B_0} \uparrow^G)$  by the set of  $p$ -regular partitions of the number  $p$ . The set bijectively corresponds to the set of indecomposable direct summand of  $k_{B_0} \uparrow^G$ . And in our setting of the present note, each such summand has a simple top (and simple socle).

For each integer  $0 \leq i \leq p-2$ , let  $P_Q(i)$  be an indecomposable direct summand of  $k_{B_0} \uparrow^G$  corresponding to the partition  $(1^{p-i}, 1, \dots, 1)$ . Then  $P_Q(i)$  has the following Lewy (and socle) structure;

$$\begin{aligned} P_Q(0) &= \begin{matrix} S_0 \\ S_1 \\ S_0 \end{matrix}, & P_Q(1) &= \begin{matrix} S_1 \\ S_0 \oplus S_2 \\ S_1 \end{matrix}, & \cdots, \\ P_Q(i) &= \begin{matrix} S_i \\ S_{i-1} \oplus S_{i+1} \\ S_i \end{matrix}, & \cdots, & P_Q(p-2) &= \begin{matrix} S_{p-2} \\ S_{p-3} \oplus S \\ S_{p-2} \end{matrix} \end{aligned} \quad (1.5)$$

where  $S_0 = k_G, S_1, \dots, S_{p-2}$  are simple  $kG$ -modules (which have  $P$  as vertex and belong to the principal block algebra  $B_0(kG)$ ). The cuspidal simple module  $S$  appears in the composition factors of  $P_Q(p-2)$ .

The complex (1.4) is isomorphic to the following complex of  $kG$ -modules;

$$\cdots \rightarrow 0 \rightarrow P_Q(p-2) \xrightarrow{\pi_{p-2}} P_Q(p-3) \xrightarrow{\pi_{p-3}} \cdots \xrightarrow{\pi_2} P_Q(1) \xrightarrow{\pi_1} P_Q(0) \xrightarrow{\pi_0} S_0 \rightarrow 0 \rightarrow \cdots \quad (1.6)$$

where the maps  $\pi_i$ 's are uniquely determined map (up to scalar) by the shapes of modules  $P_Q(i)$ 's. Thus the complex (1.6) (and (1.4)) is the first  $p-1$  terms of a  $Q$ -projective resolution of  $k_G = S_0$  and  $\Omega_Q^{p-1} k_G = \Omega_Q^{p-1} S_0 = \text{Ker } \pi_{p-2}$ . Thus

$$\Omega_Q^{p-1} k_G = \begin{matrix} S \\ S_{p-2} \end{matrix} \quad (1.7)$$

Here we denote a  $Q$ -projective syzygy of a  $kG$ -module  $V$  by  $\Omega_Q V$ . I learned from Kunugi and Miyachi that the situation above mentioned occurs.

One of our main results is the following theorem. Set  $N = N_G(P)$ . Then  $N = P \rtimes H$  and  $H \cong \mathbb{Z}_{p-1} \cong GF(p)^\times$ . Remember that we set  $A = \langle a \rangle$ .

**Theorem 1.1**  $\Omega_Q^{2(p-1)}k_G$  is an endotrivial  $kG$ -module and its Green correspondent is  $\Omega^{-2(p-1)}\Omega_A^{2(p-1)}k_N$ . Furthermore, there exists a  $kG$ -module  $M$  such that we have exact sequences of  $kG$ -modules of the following forms ;

$$0 \rightarrow k_G \rightarrow \Omega_Q^{2(p-1)}k_G \rightarrow M \rightarrow 0, \quad 0 \rightarrow \Omega^{p-2}S \rightarrow M \oplus \text{proj} \rightarrow \Omega^{p-1}S \rightarrow 0 \quad (1.8)$$

**Remark 1.2** The Green correspondent of  $\Omega_Q^{p-1}k_G$  is  $\Omega^{-(p-1)}\Omega_A^{p-1}\varepsilon_N$  and is endotrivial, where  $\varepsilon_N$  is a one dimensional  $kN$ -module with  $\varepsilon_N^2 = k_N$ . However,  $\Omega_Q^{p-1}k_G$  is not endotrivial.

#### 1.4 A vertex of $S$

$S$  is periodic because  $S \downarrow_Q$  is projective and  $Q$  is of index  $p$  in  $P$ . If we set  $G_0 := PSL(p, q) \subset G$ , then  $G/G_0$  is of order  $p$  and we see that  $S \downarrow_{G_0}$  is a direct sum of  $p$  nonisomorphic simple  $kG_0$ -modules, that is,  $S$  is induced from a  $kG_0$ -module. And then we can see that  $E$  is a minimal elementary abelian  $p$ -subgroup of  $G$  such that  $S \downarrow_E$  is not projective. By a result of Benson [5] and the fact that  $C_G(E) = E$ , we have the following proposition. Let  $K \subset L \cong SL(2, p)$  be a cyclic subgroup of  $L$  of order  $p+1$  (which is uniquely determined up to  $L$ -conjugate) and set  $N_0 = E \rtimes K \subset N_G(E)$ .

**Proposition 1.3**  $E$  is a vertex of  $S$ . And there exists an indecomposable  $kN_0$ -module  $T$  such that  $T \uparrow^G = S \oplus \text{proj}$ .

#### 1.5 The support variety $V_G(S)$ of $S$

We have  $\Omega k_E = \langle (a-1), (b-1) \rangle_{kE} = J(kE)$ . Let  $\lambda_0, \mu_0 \in H^2(E, k)$  be the Bocksteins of the following elements in  $H^1(E, k) \cong \text{Hom}_{kE}(\Omega k_E, k)$ ,

$$\begin{cases} (a-1) \mapsto 1 \\ (b-1) \mapsto 0 \end{cases}, \quad \begin{cases} (a-1) \mapsto 0 \\ (b-1) \mapsto 1 \end{cases}, \quad \text{respectively}$$

so that  $H^*(E, k) = k[\lambda_0, \mu_0] + \sqrt{0}$ . And set

$$\rho_0 = \prod_{x \in GF(p^2) - GF(p)} (\mu_0 - x\lambda_0) \in H^{2p(p-1)}(E, k)$$

**Theorem 1.4** The support variety  $V_G(S)$  of  $S$  is given by

$$V_G(S) = \text{res}_E^*(V_E(\rho_0))$$

, where  $\text{res}_E^*$  is the map from  $V_E(k) \rightarrow V_G(k)$  induced by the restriction map  $\text{res}_E : H^*(G, k) \rightarrow H^*(E, k)$ .

The first half of Section 2 is devoted to describe  $p$ -local structures of  $G$ . In the latter half of the section, we construct some cohomology elements in  $H^*(G, k)$  and some endotrivial  $kG$ -module making use of the cohomological pushout method. Proofs of Theorem 1.1 and 1.4 will be given in Section 3.

## 2 Subgroups of $G$ and $H^*(G, k)$

We shall define various subgroups of  $G = \tilde{G}/Z(\tilde{G})$  and construct some cohomology elements in  $H^*(G, k)$  which we need for our investigation below.

## 2.1 Subgroups of $\tilde{G}$

We first define subgroups of  $\tilde{G} = GL(p, q)$ . Rows and columns are indexed by the set  $GF(p) = \mathbb{Z}_p = \{0, 1, \dots, p-1\}$ . Let  $K_0 \subset GF(q)^\times$  be the multiplicative subgroup of order  $p^n$ . Let fix an element  $\zeta_0 \in K_0$  of order  $p^n$  and set  $\zeta = \zeta_0^{p^{n-1}}$  so that  $\zeta$  is of order  $p$ .

For  $\alpha_i \in GF(q)^\times$ ,  $0 \leq i \leq p-1$ , let  $d(\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{p-1})$  be the diagonal matrix with  $(i, i)$ -entry  $\alpha_i$ . And set  $c(\alpha) = d(\alpha, \alpha, \dots, \alpha)$  for  $\alpha \in GF(q)^\times$ . Set

$$\tilde{Q} = \{ d(\alpha_0, \alpha_1, \dots, \alpha_{p-1}) ; \alpha_i \in K_0 \}, \quad \tilde{Z} = \{ c(\alpha) ; \alpha \in K_0 \}$$

and

$$\tilde{D} = \{ d(\beta_0, \beta_1, \dots, \beta_{p-1}) ; \beta_i \in GF(q)^\times, (|\beta_i|, p) = 1 \}$$

Let  $\Sigma_p$  be the symmetric group on  $\{0, 1, \dots, p-1\} = \mathbb{Z}_p$ . We identify each permutation with the corresponding permutation matrix in  $\tilde{G}$ .

Let  $\tilde{a} \in \tilde{G}$  be the permutation matrix corresponding to the cyclic permutation  $(0 \ 1 \ \dots \ p-1)$ . Then

$$\begin{aligned} \tilde{a}^{-1} \cdot d(\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{p-1}) \cdot \tilde{a} &= d(\alpha_{p-1}, \alpha_0, \alpha_1, \dots, \alpha_{p-2}) \\ \{\tilde{a} \cdot d(\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{p-1})\}^p &= c(\alpha) \quad \text{where } \alpha = \alpha_0 \alpha_1 \dots \alpha_{p-1} \end{aligned} \quad (2.1)$$

Set

$$\tilde{d} = d(\zeta_0, 1, \dots, 1), \quad \tilde{b} = d(1, \zeta, \zeta^2, \dots, \zeta^{p-1}), \quad \tilde{c} = c(\zeta), \quad \tilde{u} = d(1, \zeta_1, \zeta_1^4, \dots, \zeta_1^{i^2}, \dots, \zeta_1^{(p-1)^2})$$

where  $\zeta_1 = \zeta^{\frac{1}{2}(p+1)}$  so that  $\zeta_1^2 = \zeta$ . Then we have the following equalities ;

$$(\tilde{a}\tilde{d})^p = c(\zeta_0), \quad [\tilde{a}, \tilde{b}] = \tilde{c}, \quad \tilde{u}^{-1}\tilde{a}\tilde{u} = \tilde{a}\tilde{b}\tilde{c}^{(p-1)/2}, \quad \tilde{u}^{-1}\tilde{b}\tilde{u} = \tilde{b} \quad (2.2)$$

Set  $\tilde{P} = \langle \tilde{Q}, \tilde{a} \rangle = \tilde{Q} \rtimes \langle \tilde{a} \rangle$ . Then  $\tilde{P}$  is a Sylow  $p$ -subgroup of  $\tilde{G}$ .

For each  $0 \neq s \in \mathbb{Z}_p = GF(p)$ , consider the permutation  $p(s)$  on  $\mathbb{Z}_p$  defined by

$$p(s) = \begin{pmatrix} 0 & 1 & 2 & \dots & i & \dots & p-1 \\ 0 & s & 2s & \dots & is & \dots & (p-1)s \end{pmatrix}$$

And denote by  $\tilde{h}(s)$  the corresponding permutation matrix to  $p(s)$ . Then the following equalities hold.

$$\begin{aligned} \tilde{h}(s)d(\alpha_0, \alpha_1, \dots, \alpha_i, \dots, \alpha_{p-1})\tilde{h}(s)^{-1} &= d(\alpha_0, \alpha_s, \dots, \alpha_{is}, \dots, \alpha_{(p-1)s}) \\ \tilde{h}(s)^{-1}\tilde{a}\tilde{h}(s) &= \tilde{a}^s, \quad \tilde{h}(s)^{-1}\tilde{b}\tilde{h}(s) = \tilde{b}^{s^{-1}}, \quad \tilde{h}(s)^{-1}\tilde{u}\tilde{h}(s) = \tilde{u}^{s^{-2}} \end{aligned} \quad (2.3)$$

## 2.2 Subgroups of $G$

Now we shall work in the group  $G = \tilde{G}/Z(\tilde{G}) = PGL(p, q)$ . We denote the images in  $G$  of elements and subgroups of  $\tilde{G}$  defined above by deleting  $\sim$  attached to them. Thus, for example,  $P = Q \rtimes \langle a \rangle$  is a Sylow  $p$ -subgroup of  $G$ . We also denote by  $W$  the image in  $G$  of the subgroup  $\Sigma_p$  of  $\tilde{G}$ . Let  $W_0 \cong \Sigma_{p-1}$  be the subgroup of  $W$  corresponding to the stabilizer of the point  $0 \in \{0, 1, \dots, p-1\}$  and  $W_1$  be the subgroup of  $W$  corresponding to the pointwise stabilizer of the set  $\{0, 1\}$ . Thus if we set  $H = \{ h(s) ; 0 \neq s \in \mathbb{Z}_p \}$ , then  $H \subset W_0$ .

The results we shall describe are all due to the study by Alperin and Fong [2],

### 2.2.1 $p$ -Local subgroups of $G$

By the equality (2.1), we see that  $Z(P) = \langle b \rangle$ . Again by the equality (2.1), we see that any element in  $aQ$  is of order  $p$  and is  $P$ -conjugate to  $ad^k$  for some  $k$  with  $0 \leq k \leq p-1$ . Set

$$E_k = \langle ad^k, b \rangle, \quad A_k = \langle ad^k \rangle, \quad 0 \leq k \leq p-1, \quad B = \langle b \rangle, \quad U = \langle u \rangle$$

$E_p =$  the subgroup generated by elements of order  $p$  in  $Q$

If  $p = 3$ , then  $E_p = \langle u, b \rangle$ .

By the equality (2.3),  $(ad)^{h(s)} = a^{s^{-1}}d$  and  $(a^{s^{-1}}d)^s$  is  $P$ -conjugate to  $ad^s$  by the equality (2.1). Thus the  $p-1$  subgroups  $A_k$ ,  $1 \leq k \leq p-1$  are  $P \rtimes H$ -conjugate.

**Lemma 2.1** *The following statements hold.*

1. The set  $\{E_i; 0 \leq i \leq p\}$  is a representatives set for the  $P$ -conjugacy classes of maximal elementary abelian  $p$ -subgroups of  $P$ .
2.  $E_i$ ,  $1 \leq i \leq p-1$  are  $N_G(P)$ -conjugate and the set  $\{E_0, E_1, E_p\}$  is a representatives set for the  $G$ -conjugacy classes of maximal elementary abelian  $p$ -subgroups of  $G$ .
3.  $E_0$  and  $E_1$  are of rank 2.  $E_p$  is of rank  $p-1$ .
4. Let  $G_0 \subset G$  with  $G_0 \cong PSL(p, q) \subset PGL(p, q)$ . Then  $G_0$  is a normal subgroup of  $G$  of index  $p$  and  $E_1 \not\subset G_0$ .

We are mainly concerned with the subgroup  $E_0 = \langle a, b \rangle$  so that we set  $E = E_0$ . We can write  $N_G(E) = E \rtimes L$  with  $L \cong SL(2, p)$ . We may assume that  $U \rtimes H \subset L$  where the corresponding matrices in  $SL(2, p) \cong L$  of the elements  $u, h(s)$  are given as follows (see (2.2), (2.3));

$$u \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad h(s) \mapsto \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \quad (2.4)$$

**Lemma 2.2** *The following statements hold.*

1.  $O_{p'}(N_G(Q)) = D$  and  $N_G(Q) = D \rtimes (Q \rtimes W)$ .
2.  $A \rtimes H \subset W$ . And  $W = W_1(A \rtimes H)$ ,  $W_1 \cap (A \rtimes H) = 1$ .
3.  $N_G(P) = Q \rtimes (A \rtimes H)$ .
4.  $N_G(E) = E \rtimes L$ .
5. The fusion in  $P$  is controlled by  $N_G(Q)$  and  $N_G(E)$  (and  $N_G(P)$ ).

We end this subsection with the following lemma.

**Lemma 2.3** *Set  $\mathcal{Q}_0 = \{N_G(P) \cap {}^g Q; g \in G\}$ . Then any subgroup  $R \in \mathcal{Q}_0$  is  $N_G(P)$ -conjugate to a subgroup  $Q$  or is conjugate to  $A$ .*

## 2.3 Some elements in $H^*(G, k)$

In this section, we shall construct some cohomology elements in  $H^*(G, k)$ , especially, the element  $\rho \in H^{2p(p-1)}(G, k)$  such that  $\text{res}_E \rho = \rho_0$  where  $\rho_0 \in H^{2p(p-1)}(E, k)$  is the element given in Section 1.5. The results we shall see may be known. However, we can not find appropriate literature and for the sake of completeness, we do. The study by Sasaki [14] is useful for our investigation.

### 2.3.1 $H^*(N_G(Q), k)$

We first consider  $H^*(N_G(Q), k)$ . As  $N_G(Q) = O_{p'}(N_G(Q)) \rtimes (Q \rtimes W)$  by Lemma 2.2, we may work in  $N(Q) := Q \rtimes W$ . Set

$$\tilde{Q}_1 = \{d(1, 1, \alpha_2, \dots, \alpha_{p-1}); \alpha_i \in K_0\} \subset \tilde{Q} \subset \tilde{G}, \quad \tilde{x}_1 = d(1, \zeta_0, 1, \dots, 1) \in \tilde{Q}$$

and  $Q_1 \subset G$  and  $x_1 \in G$  be the images in  $G$  of  $\tilde{Q}_1$  and  $\tilde{x}_1$ , respectively. Then  $Q = \langle x_1 \rangle \times Q_1$ .

We can write  $\Omega k_Q = \langle (x_1 - 1), J(kQ_1) \rangle_{kQ} = J(kQ)$  and consider the element in  $H^1(Q, k) = \text{Hom}_{kQ}(\Omega k_Q, k)$  satisfying

$$(x_1 - 1) \mapsto 1, \quad J(kQ_1) \mapsto 0$$

Let  $\mu \in H^2(Q, k)$  be the Bockstein of the element.  $W_1 \subset W$  normalizes  $Q_1$  and centralizes  $x_1$  because we defined  $W_1 \cong \Sigma_{p-2}$  to be the pointwise stabilizer of  $\{0, 1\}$ . Thus if we set  $N_1 = Q \rtimes W_1 \subset N(Q)$ , then  $\mu$  is canonically extended to the element in  $H^2(N_1, k)$  which we denote by the same symbol  $\mu$ .

Let  $\lambda_0, \mu_0 \in H^2(E, k)$  be the elements given in Section 1.5. And recall that we set  $B = \langle b \rangle \subset E$ .

**Lemma 2.4** *The following equalities hold.*

$$\text{res}_E \text{norm}_{N_1}^{N(Q)}(\mu) = -(\mu_0^p - \mu_0 \lambda_0^{p-1})^{p-1} = -\left\{ \prod_{y \in GF(p)} (\mu_0 - y \lambda_0) \right\}^{p-1}$$

*Proof.* Set  $\mu_1 = \text{res}_B \mu$ . We have  $bQ_1 = x_1^{p^{n-1}} Q_1$  and we see that  $\mu_1 \in H^2(B, k)$  is the Bockstein of the element in  $H^1(B, k) = \text{Hom}_{kB}(\Omega k_B, k)$  satisfying  $(b-1) \mapsto 1$ . By the equality (2.3), we have  $b^{h(s)} = b^{s^{-1}}$ . Thus  $\mu_1^{h(s)} = s \cdot \mu_1$ . Then by the Mackey formula, we have

$$\text{res}_E \text{norm}_{N_1}^{N(Q)} \mu = \text{norm}_B^E \left( \prod_{0 \neq s \in \mathbb{Z}_p} \mu_1^{h(s)} \right) = \text{norm}_B^E ((p-1)! \cdot \mu_1^{p-1}) = -\text{norm}_B^E(\mu_1)$$

Thus the first equality holds by Proposition 4.1.4 [3]. The second equality is easy to see.

The trivial  $k\Sigma_p$ -module is periodic of period  $2(p-1)$  and we have a  $2(p-1)$ -fold self extension of  $k_{N(Q)}$  of the form

$$0 \rightarrow k_{N(Q)} \rightarrow U_{2(p-1)} \rightarrow \cdots \rightarrow U_1 \rightarrow k_{N(Q)} \rightarrow 0$$

which is the first  $2(p-1)$  terms of a projective resolution of  $k_{N(Q)}$  as  $kN(Q)/Q$ -module. Let  $\chi \in H^{2(p-1)}(N(Q), k)$  be the cohomology element corresponding to the sequence. Then we may assume that

$$\text{res}_E \chi = \lambda_0^{p-1} \quad (2.5)$$

**Lemma 2.5** *Set*

$$\rho_1 = \chi^p - \text{norm}_{N_1}^{N(Q)}(\mu) \in H^{2p(p-1)}(N_G(Q), k) \quad \text{and} \quad \sigma_1 = \chi \cdot \text{norm}_{N_1}^{N(Q)}(\mu) \in H^{2(p^2-1)}(N_G(Q), k)$$

*Then*

$$\text{res}_E \rho_1 = \prod_{x \in GF(p^2) - GF(p)} (\mu_0 - x \lambda_0), \quad \text{res}_E \sigma_1 = \left\{ \lambda_0 \prod_{y \in GF(p)} (\mu_0 - y \lambda_0) \right\}^{p-1}$$

*Proof.* A proof of Lemma 4.2 [14] works well by the equality (2.5) and Lemma 2.4.

### 2.3.2 $\rho \in H^{2p(p-1)}(G, k)$ with $\text{res}_E \rho = \rho_0$

By Lemma 2.5,  $\text{res}_E \rho_1 \in H^{2p(p-1)}(E, k)$  and  $\text{res}_E \sigma_1 \in H^{2(p^2-1)}(E, k)$  are invariant under the action of  $GL(E) = \text{Aut} E$ . They are the so called Dickson invariants (see Section 8.1, 8.2 [4]). In particular, they are  $N_G(E)$ -invariant. Thus by Lemma 2.2.5, we have elements  $\rho \in H^{2p(p-1)}(G, k)$  and  $\sigma \in H^{2(p^2-1)}(G, k)$  such that

$$\text{res}_E \rho = \text{res}_E \rho_1, \quad \text{res}_E \sigma = \text{res}_E \sigma_1 \quad (2.6)$$

Thus  $\text{res}_E \rho = \rho_0$ .

## 2.4 Some endotrivial $kG$ -module

Let  $L_\rho$  be the Carlson module of  $\rho$ . We see that  $\text{res}_B \rho \in H^{2p(p-1)}(B, k) = H^{2p(p-1)}(Z(P), k)$  is not nilpotent and we can apply the method of constructing endotrivial modules (see [8], [10]). All the results below are due to Carlson [8].

We are concerned with the group  $E$ . The variety  $V_G(L_\rho)$  decomposes as

$$V_G(L_\rho) = V_0 \cup V'_0 \text{ with } V_0 \cap V'_0 = \{0\} \text{ where } V_0 = \text{res}_E^*(V_E(L_\rho)) = \text{res}_E^*(V_E(L_{\rho_0}))$$

Here  $L_{\rho_0}$  is the Carlson module of  $\rho_0$ . Then  $L_\rho$  decomposes as

$$L_\rho = L_0 \oplus L'_0 \text{ where } V_G(L_0) = V_0, \quad V_G(L'_0) = V'_0$$

Now set  $Y = \Omega^{2p(p-1)} k_G / L'_0$ .  $Y$  is an endotrivial  $kG$ -module which appears as a pushout in the following diagram ;

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & L'_0 & \xlongequal{\quad} & L'_0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_\rho & \longrightarrow & \Omega^{2p(p-1)} k_G & \longrightarrow & k_G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & L_0 & \longrightarrow & Y & \longrightarrow & k_G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

For our discussion, the dual  $X = Y^*$  of  $Y$  is convenient to use. Set  $M_0 = L_0^*$  so that we have an exact sequence of the form

$$0 \rightarrow k_G \rightarrow X \rightarrow M_0 \rightarrow 0 \text{ and } V_G(M_0) = \text{res}_E^*(V_E(L_{\rho_0})) \quad (2.7)$$

The endotrivial module  $X$  satisfies the following ;

$$X \downarrow_E = \Omega^{-2p(p-1)} k_E \oplus \text{proj}, \quad X \downarrow_{E_i} = k_{E_i} \oplus \text{proj}, \text{ for } 1 \leq i \leq p \quad (2.8)$$

By the construction of  $Y$ , if we set  $N_p = B \rtimes H$ , then  $Y \downarrow_{N_p} = k_{N_p} \oplus \text{proj}$ .  $N_p$  and  $N_0 = A \rtimes H$  are conjugate in  $N_G(E)$  (see (2.4)). Thus we have

$$X \downarrow_{N_0} = k_{N_0} \oplus \text{proj} \quad (2.9)$$

## 3 Proofs of Theorem 1.1 and 1.4

In this section, we shall give proofs of theorems in Section 1.

### 3.1 A proof of Theorem 1.1

#### 3.1.1 $\Omega_Q^{2(p-1)} k_G$

First we shall construct the first  $p-1$  terms of a  $Q$ -projective resolutions of  $\Omega_Q^{p-1} k_G = \frac{S}{S_{p-2}}$ . Simple  $kG$ -modules  $S_i$ 's and  $S$  are all self-dual. Taking the dual of the  $Q$ -projective resolution (1.6) of  $k_G$ , we



have a  $Q$ -injective resolution

$$0 \rightarrow k_G \xrightarrow{\pi_0^*} P_Q(0) \xrightarrow{\pi_1^*} P_Q(1) \rightarrow \cdots \xrightarrow{\pi_{p-3}^*} P_Q(p-3) \xrightarrow{\pi_{p-2}^*} P_Q(p-2) \xrightarrow{g} \frac{S_{p-2}}{S} \rightarrow 0 \quad (3.1)$$

which is a  $Q$ -projective resolution of  $(\Omega_Q^{p-1} k_G)^* = \frac{S_{p-2}}{S}$ . As  $S \downarrow_Q$  is projective, we see that the exact sequence  $0 \rightarrow S \rightarrow \frac{S_{p-2}}{S} \rightarrow S_{p-2} \rightarrow 0$  is  $Q$ -split and a  $Q$ -projective resolution of  $S$  is a usual projective resolution. Thus by the sequence (3.1) the first  $p-1$  terms of a  $Q$ -projective resolution of  $S_{p-2}$  has the form

$$0 \rightarrow \Omega_Q^{p-1} S_{p-2} \xrightarrow{g_{p-1}} P'_Q(0) \xrightarrow{g_{p-2}} P'_Q(1) \rightarrow \cdots \rightarrow P'_Q(p-3) \xrightarrow{g_1} P_Q(p-2) \xrightarrow{g_0} S_{p-2} \rightarrow 0 \quad (3.2)$$

where  $P'_Q(i) = P_Q(i) \oplus \text{proj}$ . Furthermore, we have an exact sequence of the following form ;

$$0 \rightarrow \Omega^{p-1} S \rightarrow k_G \oplus \text{proj} \rightarrow \Omega_Q^{p-1} S_{p-2} \rightarrow 0 \quad (3.3)$$

We also have the  $Q$ -split exact sequence  $0 \rightarrow S_{p-2} \rightarrow \Omega_Q^{p-1} k_G \rightarrow S \rightarrow 0$  and by the same argument as above, we have the followings. By the sequence (3.2), the first  $p-1$  terms of a  $Q$ -projective resolution of  $\Omega_Q^{p-1} k_G$  has the form

$$0 \rightarrow \Omega_Q^{2(p-1)} k_G \xrightarrow{f_{p-1}} P''_Q(0) \xrightarrow{f_{p-2}} P''_Q(1) \rightarrow \cdots \rightarrow P''_Q(p-3) \xrightarrow{f_1} P''_Q(p-2) \xrightarrow{f_0} \Omega_Q^{p-1} k_G \rightarrow 0 \quad (3.4)$$

where  $P''_Q(i) = P_Q(i) \oplus \text{proj}$ . Furthermore, we have an exact sequence of the following form ;

$$0 \rightarrow \Omega_Q^{p-1} S_{p-2} \rightarrow \Omega_Q^{2(p-1)} k_G \oplus \text{proj} \rightarrow \Omega^{p-1} S \rightarrow 0 \quad (3.5)$$

By (3.3), we have an exact sequence of the form  $0 \rightarrow k_G \rightarrow \Omega_Q^{p-1} S_{p-2} \rightarrow \Omega^{p-2} S \rightarrow 0$ . And there exists a  $kG$ -module  $M$  such that we have exact sequences of the form ;

$$0 \rightarrow k_G \rightarrow \Omega_Q^{2(p-1)} k_G \rightarrow M \rightarrow 0, \quad 0 \rightarrow \Omega^{p-2} S \rightarrow M \oplus \text{proj} \rightarrow \Omega^{p-1} S \rightarrow 0$$

Thus the second statement in Theorem 1.1 follows.

### 3.1.2 $\Omega_Q^{2(p-1)} k_G \downarrow_{N_G(P)}$

In this section, we investigate the restriction of  $\Omega_Q^{2(p-1)} k_G$  to the subgroup  $N_G(P)$ . We refer the article by Bouc [6] for general results of relative syzygies.

Set  $N = N_G(P)$  and  $\mathcal{Q}_0 = \{ N \cap {}^g Q ; g \in G \}$ . And set  $\mathcal{Q}'_0 = \{ Q, A \}$ . By Lemma 2.3, for  $kN$ -modules,  $\mathcal{Q}_0$ -projective covers coincide with  $\mathcal{Q}'_0$ -projective covers.

$N = P \rtimes H$  and  $N/Q \cong A \rtimes H$ . Let  $N \rightarrow N/P = H$  be the canonical group surjection. The map  $H \rightarrow GF(p)^\times$ ,  $h(s) \mapsto s$  is a group homomorphism (actually, isomorphism). Let  $\varphi_N : N \rightarrow GF(p)^\times$  be the composite of these two maps and we denote by the same symbol  $\varphi_N$  the corresponding one dimensional  $kN$ -module. Then by the equality (2.3), we can see that

$$\Omega_Q^2 k_N = \varphi_N \quad (3.6)$$

Taking relative syzygies is compatible with the restriction to subgroups and the followings hold ;

$$\Omega_Q k_G \downarrow_N \equiv \Omega_{\mathcal{Q}_0} k_N = \Omega_{\mathcal{Q}'_0} k_N \pmod{\mathcal{Q}_0}$$

By the fact that  $Q \cap A = 1$ , and by a result of Thévenaz and Bouc (Lemma 5.2.1 [6], see also an argument by Alperin [1]), we have

$$\Omega \Omega_{\mathcal{Q}'_0} k_N = \Omega_Q \Omega_A k_N$$

Thus by the commutativity of taking relative syzygies,

$$\Omega_Q k_G \downarrow_N \equiv \Omega^{-1} \Omega_Q \Omega_A k_N \pmod{\mathcal{Q}_0}, \quad \Omega_Q^2 k_G \downarrow_N \equiv \Omega^{-2} \Omega_A^2 \varphi_N \pmod{\mathcal{Q}_0}$$

where we used the equality (3.6).

Thus for any even integer  $2m$ , a Green correspondent of  $\Omega_Q^{2m} k_G$  is  $\Omega^{-2m} \Omega_A^{2m} \varphi_N^m$  and is an endotrivial  $kN$ -module (Proposition 4.2 [10]). For  $m = \frac{1}{2}(p-1)$ , the dimension of  $\Omega_Q^{2m} k_G = \Omega_Q^{p-1} k_G$  is  $q^{\frac{1}{2}p(p-1)}$ , the degree of the Steinberg character. We see that  $q^{\frac{1}{2}p(p-1)} - 1$  is divisible by  $p^{n+1}$  but is not divisible by  $p^{n+2}$ . Thus  $\Omega_Q^{p-1} k_G$  itself is not endotrivial and Remark 1.2 follows.

A Green correspondent of  $\Omega_Q^{2(p-1)} k_G$  is  $\Omega^{-2(p-1)} \Omega_A^{2(p-1)} \varphi_N^{p-1} = \Omega^{-2(p-1)} \Omega_A^{2(p-1)} k_N$ . The sequences (1.6) and (3.4) are  $\mathcal{Q}$ -split and therefore we have

$$\Omega_Q^{2(p-1)} k_G + \sum_{i=0}^{p-2} (-1)^{p-2-i} P_Q(i)'' = \Omega_Q^{p-1} k_G = k_G + \sum_{i=0}^{p-2} (-1)^{p-2-i} P_Q(i) \quad (3.7)$$

in the Green ring (the representation ring) of  $kQ$ -modules. As the sequences are sequences of  $kG$ -modules, the equality (3.7) holds in the Green ring of  $kR$ -modules for any  $R \in \mathcal{Q}_0$ . Thus  $\Omega_Q^{2(p-1)} k_G \downarrow_R = k_R \oplus \text{proj.}$  We can write as

$$\Omega_Q^{2(p-1)} k_G \downarrow_N = \Omega^{-2(p-1)} \Omega_A^{2(p-1)} k_N \oplus V$$

where  $V$  is a  $\mathcal{Q}_0$ -projective  $kN$ -module. And then we can conclude that  $V$  is projective and a proof of Theorem 1.1 is completed.

### 3.2 A proof of Theorem 1.4

#### 3.2.1 $\Omega_Q^{2(p-1)} k_G = X$

We shall show that  $\Omega_Q^{2(p-1)} k_G \cong X$  where  $X$  is the endotrivial  $kG$ -module given in Section 2.4.

We saw that  $\Omega_Q^{2(p-1)} k_G$  is endotrivial. In the group  $N = N_G(P)$ , any conjugate of  $A$  intersects trivially with  $E_i$ ,  $i \neq 0$ . Thus as endotrivial  $kN$ -modules,  $\Omega_Q^{2(p-1)} k_G$  and the module  $X$  have the same "type" by the equality (2.8). We can see that the equality (3.7) holds in the Green ring of  $kN_0$  where  $N_0 = A \rtimes H$  because  $A \in \mathcal{Q}_0$ . Thus  $\Omega_Q^{2(p-1)} k_G \downarrow_{N_0} = k_{N_0} \oplus \text{proj.}$  Then by the equality (2.9), we see that Green correspondents of  $\Omega_Q^{2(p-1)} k_G$  and  $X$  are isomorphic and the result follows.

#### 3.2.2 $V_G(S)$

We refer to Benson's book [3] for the support variety of modules.

Let  $0 \rightarrow k_G \xrightarrow{f} \Omega_Q^{2(p-1)} k_G \rightarrow M \rightarrow 0$  be the first exact sequence given in (1.8). By the second exact sequence in (1.8),  $M$  is a periodic module. Thus  $f \downarrow_E$  is a not projective map because if it were, then  $M \downarrow_E$  would have a direct summand isomorphic to  $\Omega^{-1} k_E$ , a contradiction.

Consider the restriction of the sequence to  $E$ . We saw that  $\Omega_Q^{2(p-1)} k_G \cong X$  and therefore we have  $\Omega_Q^{2(p-1)} k_G \downarrow_E = \Omega^{-2p(p-1)} k_E$ . Thus the exact sequence which we consider has the form ;

$$0 \rightarrow k_E \xrightarrow{f_0} \Omega^{-2p(p-1)} k_E \rightarrow M' \rightarrow 0 \quad (3.8)$$

where  $M'$  is a direct summand of  $M \downarrow_E$ .

We have an isomorphism  $\text{Hom}_{kE}(k_E, \Omega^{-2p(p-1)} k_E) \cong H^{2p(p-1)}(E, k) = k[\lambda_0, \mu_0] + \sqrt{0}$ , where  $\lambda_0, \mu_0 \in H^2(E, k)$  is the cohomology elements given in Section 1.5. The corresponding elements  $\nu \in H^{2p(p-1)}(E, k)$  to  $f_0$  under the isomorphism is  $N_G(E)$ -invariant. We see that  $H^*(E, k)^{N_G(E)} = k[\lambda_0, \mu_0]^{N_G(E)} + \sqrt{0}^{N_G(E)}$ . By that fact that  $N_G(E)/E \cong SL(2, p)$ ,  $k[\lambda_0, \mu_0]^{N_G(E)}$  is generated by  $\rho_0 \in H^{2p(p-1)}(E, k)$  and  $\sigma'_0 = \lambda_0^p \mu_0 - \lambda_0 \mu_0^p \in H^{2(p+1)}(E, k)$  (see Section 8.2 [4]). Thus  $\nu \equiv \rho_0 \pmod{\sqrt{0}}$ .  $M' = L_\nu^*$  where  $L_\nu$  is the Carlson module of  $\nu$ . Thus  $V_E(M') = V_E(\nu) = V_E(\rho_0)$ .

Again by the second exact sequence in (1.8),  $V_E(\rho_0) = V_E(M') \subset V_E(S)$ . As  $S$  is  $E$ -projective and periodic, we can conclude that  $V_G(S) = \text{res}_E^*(V_E(\rho_0))$  and a proof of Theorem 1.4 is completed.

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